COMMENTS ON VALUATIONS ASSOCIATED TO SYSTEMS OF VERTICES/EDGES AND THE MAIN THEOREM OF POP-STIX

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Let k be an *arbitrary complete discrete valuation field of mixed characteristic* whose residue characteristic we denote by p, $\overline{k}$ an algebraic closure of k, $G_k \stackrel{\text{def}}{=} \text{Gal}(\overline{k}/k)$, Σ a *set of primes* that contains a prime $l \neq p$, X a *proper hyperbolic curve* over k. Suppose, further, that k is l*-cyclotomically full*, i.e., that the image of the l-adic cyclotomic character $G_k \to \mathbb{Z}_l^\times$ is *open* in $\mathbb{Z}_l^\times$. Write

$$\Pi_X \twoheadrightarrow \Pi_X^{(\Sigma)}$$

for the *geometrically pro-*Σ *quotient* of the étale fundamental group Π_X of X. Thus, we have a natural surjection $\Pi_X^{(\Sigma)} \twoheadrightarrow G_k$. Let

$$\ldots \ \to \ X_{i+1} \ \to \ X_i \ \to \ \ldots$$

[where i ranges over the positive integers] be a cofinal system of *finite étale connected Galois coverings of* X *with stable reduction* arising from open subgroups of $\Pi_X^{(\Sigma)}$ and

$$s : G_k \to \Pi_X^{(\Sigma)}$$

a *section* of $\Pi_X^{(\Sigma)} \twoheadrightarrow G_k$. Then in the "Comments on a Combinatorial Version of the Section Conjecture and the Main Theorem of Pop-Stix" dated March 3, 2011 (cf. [CbSC], (5)), we showed that

($*^{v/e}$) [after possibly passing to a cofinal subsystem of the given system of coverings] there exists *either* a [not necessarily unique] *system of vertices*

$$\ldots \ \rightsquigarrow \ v_{i+1} \ \rightsquigarrow \ v_i \ \rightsquigarrow \ \ldots$$

or a [not necessarily unique] *system of edges*

$$\ldots \ \rightsquigarrow \ e_{i+1} \ \rightsquigarrow \ e_i \ \rightsquigarrow \ \ldots$$

— i.e., each v_i (respectively, e_i) is an irreducible component (respectively, node) of the special fiber of the stable model $\mathcal{X}_i$ of X_i that is *fixed* by the natural action of the image $\text{Im}(s)$ of the section s; the image of the irreducible component v_{i+1} (respectively, node e_{i+1}) in $\mathcal{X}_i$ is contained in the irreducible component v_i (respectively, node e_i).

In the present note, we verify (cf. (1), (2) below), by means of a quite elementary argument in scheme theory/commutative algebra, that

($*^{\mathrm{val}}$) such a system of vertices or edges determines a *system of valuations* of the function fields K_i of the X_i that are *fixed* by the natural action of $\mathrm{Im}(s)$.

In particular, we obtain a proof of the *main theorem of Pop-Stix* (cf. [PS]) by means of elementary graph-theoretic and scheme-/ring-theoretic considerations, without resorting to the use of *highly nontrivial arithmetic* results such as Tamagawa's *"resolution of nonsingularities"* [i.e., the main result of [Tama]]. Here, we recall that this result of [Tama] depends, in an essential way, on *highly arithmetic arguments* that require one to take Σ to be the *set of all primes*, as well as on relatively deep *wild ramification* properties of p-power coverings of X. In particular, the essential role played by this result in the proof of [PS] has the effect of *portraying the phenomenon discussed in the main theorem of [PS] as being a consequence of such deep arithmetic considerations.* In fact, however, the arguments of the present note imply that

the essential phenomenon discussed in the main theorem of [PS] is [not "arithmetic" or "p-adic", but rather] *"l-adic"* and *"combinatorial"* in nature and may be obtained as a consequence of *quite elementary considerations* concerning *finite group actions on graphs* and *scheme theory/commutative algebra.*

(1) Suppose that one has a *system of vertices* $\{v_i\}$ as in ($*^{v/e}$). If [after possibly passing to a cofinal subsystem of the given system of coverings] each v_{i+1} maps *quasi-finitely* to v_i, then the system of valuations associated to the v_i already yields a system of valuations as desired. Thus, [after possibly passing to a cofinal subsystem of the given system of coverings] we may assume without loss of generality that v_{i+1} maps to a *closed point* x_i of v_i. If [after possibly passing to a cofinal subsystem of the given system of coverings] the x_i are all nodes, then we obtain a *system of edges* $\{e_i\}$ as in ($*^{v/e}$); this situation will be dealt with in (2) below. Thus, [after possibly passing to a cofinal subsystem of the given system of coverings] we may assume without loss of generality that each x_i is a *smooth point.* In particular, the local ring R_i of $\mathcal{X}_i$ at x_i is *regular of dimension* 2, hence a *UFD.* Write

$$\mathrm{ord}_i : K_i^\times \to \mathbb{Q}$$

for the *valuation* associated to v_i, normalized so as to restrict to a fixed [i.e., independent of i], given valuation on k. Then it follows immediately from the definition of x_i, together with the fact that R_i is a *UFD*, that we have

$$\mathrm{ord}_{j'}(f) \;\geq\; \mathrm{ord}_j(f) \;\geq\; 0$$

for any nonzero $f \in R_i \subseteq K_i$, $j' \geq j \geq i$. [Here, we think of the various K_i as being related to one another via the natural inclusions $K_i \subseteq \ldots \subseteq K_j \subseteq \ldots \subseteq K_{j'}$.] Next, let us observe that it follows immediately from the fact that each $\text{ord}_j(-)$ is a *valuation* that, if we set $\text{ord}_j(0) \stackrel{\text{def}}{=} +\infty$, then the subset

$$R_i \supseteq I_i \stackrel{\text{def}}{=} \{f \in R_i \mid \lim_{j \to \infty} \text{ord}_j(f) = +\infty\}$$

is, in fact, a *prime ideal* of R_i whose intersection with the ring of integers $\mathcal{O}_k \subseteq R_i$ of k is equal to $\{0\}$. In particular, the *height* of I_i is ≤ 1. If [after possibly passing to a cofinal subsystem of the given system of coverings] the I_i are all of height 1, then it follows immediately that I_i determines a *closed point* ξ_i of X_i, and that the system of valuations associated to the ξ_i yields a system of valuations as desired [indeed, of the "ideal type", from the point of view of the original Section Conjecture!]. Thus, [after possibly passing to a cofinal subsystem of the given system of coverings] we may assume without loss of generality that each I_i is of height 0, hence equal to $\{0\}$. But this implies that, for $f \in K_i^{\times}$, the quantity

$$\text{ord}_{\infty}(f) \stackrel{\text{def}}{=} \lim_{j \to \infty} \text{ord}_j(f) \in \mathbb{R}$$

is *well-defined*. Moreover, one verifies immediately that $\text{ord}_{\infty}(-)$ determines a *valuation* on K_i that is *fixed* by the action of $\text{Im}(s)$. In particular, one obtains a system of valuations as desired.

(2) Suppose that one has a *system of edges* $\{e_i\}$ as in $(*^{v/e})$. Write $\mathcal{X}_i^{\log}$ for the *regular log scheme* whose underlying scheme is $\mathcal{X}$ and whose interior is the generic fiber $X_i \subseteq \mathcal{X}_i$. Thus, the characteristic of the log structure of $\mathcal{X}_i^{\log}$ at x_i determines — by tensoring the groupification of the characteristic with $\mathbb{R}$ — a 2-dimensional real vector space, whose *dual* we denote by M_i. Thus, M_i is equipped with a natural *positive rational structure* P_i [i.e., a submonoid isomorphic to $\mathbb{Q}_{\geq 0} \oplus \mathbb{Q}_{\geq 0}$ that generates M_i as a real vector space]. [Put another way, M_i is the sort of real vector space that appears in discussions of *toric varieties*.] The natural morphism $\mathcal{X}_{i+1}^{\log} \to \mathcal{X}_i^{\log}$ induces an $\mathbb{R}$-*linear map of vector spaces* $M_{i+1} \to M_i$ of *rank* ≥ 1 that maps P_{i+1} into P_i. Write $\overline{P}_i \subseteq M_i$ for the closure of P_i in M_i. Let us refer to as a $\overline{P}$-*ray* of M_i a ray of M_i emanating from the origin that is contained in $\overline{P}_i$. Now it follows immediately from the *compactness* of the space of $\overline{P}$-rays of M_i that [after possibly passing to a cofinal subsystem of the given system of coverings] we may assume that there exists a *compatible system* $\{\lambda_i\}$ of $\overline{P}$-rays of the M_i which are, moreover, *fixed* by the action of $\text{Im}(s)$. Suppose that [after possibly passing to a cofinal subsystem of the given system of coverings] each λ_i is *rational* [i.e., generated by an element of P_i]. Then λ_i corresponds to an *irreducible component* v_i of a suitable blow-up of $\mathcal{X}_i$ at e_i; one may construct these blow-ups so that v_{i+1} maps into v_i. If [after possibly passing to a cofinal subsystem of the given system of coverings] each v_{i+1} maps *quasi-finitely* to v_i, then the system of valuations associated to the v_i already yields a system of valuations as desired. Thus, [after possibly passing to a cofinal subsystem of the given system of coverings] we may assume without loss of generality that v_{i+1} maps to a *closed point* x_i of v_i; moreover, it follows immediately from the fact that the λ_i form a compatible system that each

x_i is a *smooth point*. Thus, one may construct either a system of *closed points* ξ_i of X_i or a system of *"limit valuations* $\mathrm{ord}_\infty(-)$*"* as in (1); this yields a system of valuations as desired. This completes the proof in the case where the λ_i are rational. Thus, [after possibly passing to a cofinal subsystem of the given system of coverings] we may assume without loss of generality that each λ_i is *irrational*. But then it is well-known that each λ_i determines a valuation on K_i; the compatibility of these valuations as one varies i follows immediately from the compatibility of the λ_i. Thus, one obtains a system of valuations as desired.

(3) The present note benefited from discussions with Fumiharu Kato in November 2010.

Bibliography

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